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LETTER TO THE EDITOR

Exact random walk distributions using noncommutative geometry

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Abstract. Using the results obtained by the non-commutative geometry techniques applied to the Harper equation, we derive the areas distribution of random walks of length N on a two-dimensional square lattice for large N, taking into account finite-size contributions.

Let us consider on a square lattice all closed paths of length N starting at the origin. For such a path Γ, let $A(\Gamma)$ be its algebraic area. As $N \to \infty$, the average size of such a path increases as $\sqrt{N}$ so that $A(\Gamma) \simeq N$ and the renormalized area will be $a = A/N$. We want to compute the probability distribution $\mathcal{P}(A, N)$ of the areas at large but finite N.

In the limit $N \to \infty$, the distribution was computed first by Lévy using Brownian paths [1]. We will give a method based upon the Harper model, allowing the computation of finite-size corrections in a systematic way. The Harper model was designed in 1955 [2] as the simplest non-trivial model describing the motion of an electron sitting on a two-dimensional square lattice and submitted to a uniform magnetic field. Let ϕ be the magnetic flux through the unit cell and let $\phi_0 = h/e$ be the quantum flux. We set $\gamma = 2\pi\phi/\phi_0$. Then given $m = (m_1, m_2) \in \mathbb{Z}^2$, we denote by $W(m)$ the corresponding magnetic translations [3]. They satisfy the Weyl commutation rules

$$W(m)W(m') = W(m+m')\exp\left(\mathrm{i}\frac{\gamma}{2} m \wedge m'\right) \tag{1}$$

where $m' \wedge m = m'_1 m_2 - m'_2 m_1 \in \mathbb{Z}$. Note that γ plays a rôle similar to the Planck constant in the canonical commutation relation.

Harper's model is given by the following Hamiltonian:

$$H = \sum_{|a|=1} W(a) \tag{2}$$

where $|a| = |a_1| + |a_2|$ if $a = (a_1, a_2) \in \mathbb{Z}^2$. In addition, one defines the trace per unit area as the unique linear map $\mathcal{T}$ on the algebra generated by the $W(m)$s such that

$$\mathcal{T}(W(m)) = \delta_{m,0}. \tag{3}$$

Then from (1)–(3), we obtain:

$$\mathcal{T}(H^N) = \sum_{\Gamma:\text{closed paths of length } N} \mathrm{e}^{\mathrm{i}\gamma A(\Gamma)/2}$$

where the sum is taken on the set of closed paths starting at the origin of length N. Note that N should be even to get a non-zero sum. Let Ω_N be the number of such closed paths, we then obtain:

$$\sum_{A=-A_{\max}}^{A_{\max}} \mathcal{P}_N(A/N)\exp(\mathrm{i}xA/N) = \Omega_N^{-1}\sum_{\Gamma}\exp(\mathrm{i}xA(\Gamma)/N) = \Omega_N^{-1}\mathcal{T}(H^N)|_{\gamma=x/N}. \tag{4}$$

From this relation we obtain:

$$\Omega_N = \mathcal{T}(H^N)|_{\gamma=0} = \int \frac{\mathrm{d}k_1\,\mathrm{d}k_2}{4\pi^2}(2\cos k_1 + 2\cos k_2)^N$$
$$= \frac{4^{N+1}}{2\pi N}(1+\mathrm{O}(1/N^2)) \qquad \text{as } N\to\infty. \tag{5}$$

Moreover as $N\to\infty$, for a given value of x, $\gamma = x/N$ tends to zero, so that we can use a semiclassical argument to compute $\mathcal{T}(H^N)$. It has been shown that the spectrum of H is made of Landau sublevels [4]:

$$E_\ell^{\pm}(\gamma) = \pm\left(4-\gamma(2\ell+1)+\frac{\gamma^2}{16}[1+(2\ell+1)^2]-\mathrm{O}(\gamma^3)\right) \qquad \ell = 0,1,\ldots,\mathrm{O}(1/\gamma) \tag{6}$$

each with multiplicity per unit area $g_\ell^{\pm} = \gamma/2\pi$. So,

$$\mathcal{T}(H^N) = \sum_{\pm}\sum_{\ell}(E_\ell^{\pm}(\gamma))^N[\gamma/(2\pi)]. \tag{7}$$

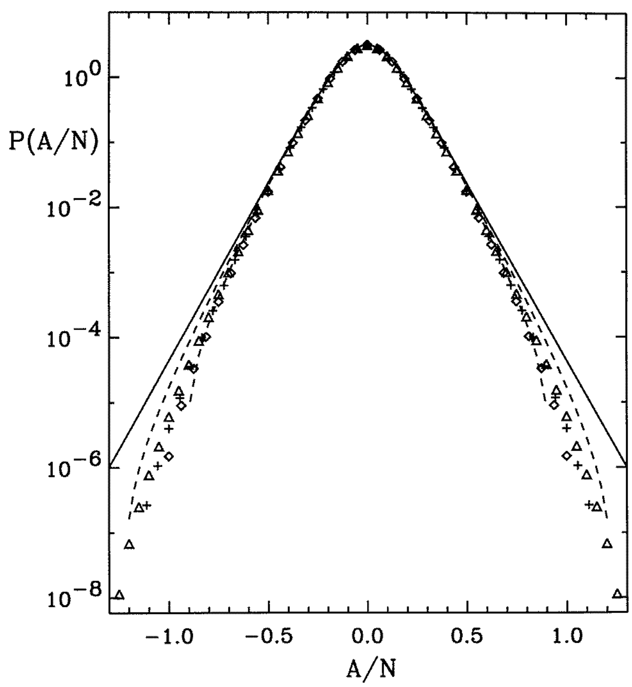

Figure 1. Scaling function for the probability of having a loop of area A for a random walk of N steps. Open diamond, plus and open triangle symbols correspond to the finite-size data for $N = 16$, 18 and 20, respectively. The full curve corresponds to the universal function (9), as $N\to\infty$; whereas the broken curves include the $1/N$ correction term for $N = 20$ and 40.

This gives

$$\mathcal{T}(H^N) = \frac{4^{N+1}}{2\pi N}\frac{x/4}{\sinh(x/4)}\left[1 - \frac{1}{2N}\frac{(x/4)^2}{\sinh^2(x/4)} + \mathrm{O}(1/N^2)\right]. \tag{8}$$

Using (4), we obtain the probability distribution

$$\mathcal{P}_N(a) = \frac{\pi}{\cosh^2(2\pi a)} + \mathrm{O}(1/N). \tag{9}$$

We numerically computed $\mathcal{P}_N(a)$ from formula (8) and compared the result with exact numerical calculations (figure 1).

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